# Complex note

Created by Mr. Francis Hung on 20211020 Last updated: 2021-10-20

Text book Algebra p.218 Q6

Express $\dfrac{\sin 9\theta}{\sin\theta}$ as a polynomial in cos θ.

Hence deduce that $\sec^2\dfrac{\pi}{9}+\sec^2\dfrac{2\pi}{9}+\sec^2\dfrac{4\pi}{9}=36$ and $\sec\dfrac{\pi}{9}\sec\dfrac{2\pi}{9}\sec\dfrac{4\pi}{9}=8$.

Let $z = \cos\theta + i\sin\theta = c + is$, where $c = \cos\theta$, $s = \sin\theta$

$z^9 = \cos 9\theta + i\sin 9\theta = (c + is)^9$

Compare the imaginary parts of both sides:

$\sin 9\theta = 9c^8s - 84c^6s^3 + 126c^4s^5 - 36c^2s^7 + s^9$

$\dfrac{\sin 9\theta}{\sin\theta} = 9c^8 - 84c^6(1 - c^2) + 126c^4(1 - c^2)^2 - 36c^2(1 - c^2)^3 + (1 - c^2)^4$

$= 9c^8 - 84c^6 + 84c^8 + 126c^4(1 - 2c^2 + c^4) - 36c^2(1 - 3c^2 + 3c^4 - c^6) + (1 - 4c^2 + 6c^4 - 4c^6 + c^8)$

$= 256\cos^8\theta - 448\cos^6\theta + 240\cos^4\theta - 40\cos^2\theta + 1$

Let $\dfrac{\sin 9\theta}{\sin\theta} = 0$, then $9\theta = m\pi$ and $\sin\theta \neq 0$

$\theta = \dfrac{m\pi}{9}$, $m = 1, 2, 3, 4, 5, 6, 7, 8$

$c = \cos\dfrac{m\pi}{9}$ is a root of $256c^8 - 448c^6 + 240c^4 - 40c^2 + 1 = 0$

Note that $\cos\dfrac{8\pi}{9} = -\cos\dfrac{\pi}{9}$, $\cos\dfrac{7\pi}{9} = -\cos\dfrac{2\pi}{9}$, $\cos\dfrac{6\pi}{9} = -\cos\dfrac{3\pi}{9}$, $\cos\dfrac{5\pi}{9} = -\cos\dfrac{4\pi}{9}$

$\therefore \cos^2\dfrac{m\pi}{9}$ $(m = 1, 2, 3, 4)$ are roots of $256x^4 - 448x^3 + 240x^2 - 40x + 1 = 0$

Consider the transformation $y = \dfrac{1}{x}$,

$\sec^2\dfrac{m\pi}{9}$ $(m = 1, 2, 3, 4)$ are roots of $\dfrac{256}{y^4} - \dfrac{448}{y^3} + \dfrac{240}{y^2} - \dfrac{40}{y} + 1 = 0$

i.e. $y^4 - 40y^3 + 240y^2 - 448y + 256 = 0$

In particular, $m = 3$, $\sec^2\dfrac{m\pi}{9} = \sec^2\dfrac{3\pi}{9} = \sec^2\dfrac{\pi}{3} = 4$

$\sec^2\dfrac{\pi}{9}+\sec^2\dfrac{2\pi}{9}+4+\sec^2\dfrac{4\pi}{9}$ = sum of roots = 40

$\therefore \sec^2\dfrac{\pi}{9}+\sec^2\dfrac{2\pi}{9}+\sec^2\dfrac{4\pi}{9}=36$

$\sec^2\dfrac{\pi}{9}\sec^2\dfrac{2\pi}{9}\cdot 4\cdot\sec^2\dfrac{4\pi}{9}$ = product of roots = 256

$\sec\dfrac{\pi}{9}\sec\dfrac{2\pi}{9}\sec\dfrac{4\pi}{9} = 8$

Prove that the roots of $x^3+\sqrt{7}x^2-7x+\sqrt{7}=0$ are $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, $\tan\frac{4\pi}{7}$.

Let $c = \cos\theta$, $s = \sin\theta$, $z = c + is$

$z^7 = (c + is)^7$

$\cos 7\theta + i\sin 7\theta = c^7 + 7i\,c^6s - 21\,c^5s^2 - 35ic^4s^3 + 35c^3c^4 + 21ic^2s^5 - 7cs^6 - is^7$

Compare the real aprts and the imaginary parts,

$\cos 7\theta = c^7 - 21\,c^5s^2 + 35c^3c^4 - 7cs^6$

$\sin 7\theta = 7c^6s - 35c^4s^3 + 21c^2s^5 - s^7$

Consider $\tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta} = \frac{7c^6s-35c^4s^3+21c^2s^5-s^7}{c^7-21c^5s^2+35c^3s^4-7cs^6}$

Let $t = \tan\theta$; divide the numerator and the denominator by $c^7$.

$\tan 7\theta = \frac{7t-35t^3+21t^5-t^7}{1-21t^2+35t^4-7t^6}$

Let $\tan 7\theta = 0 \Rightarrow \theta = \frac{m\pi}{7}$, where $m \in \mathbf{Z}$

$7t - 35t^3 + 21t^5 - t^7 = 0$

$t(t^6 - 21t^4 + 35t^2 - 7) = 0$, the roots are $0, \pm\tan\frac{\pi}{7}, \pm\tan\frac{2\pi}{7}, \pm\tan\frac{3\pi}{7}$.

$\because \tan\frac{4\pi}{7} = \tan(\pi - \frac{3\pi}{7}) = -\tan\frac{3\pi}{7}$

$\therefore$ The roots of $t^6 - 21t^4 + 35t^2 - 7 = 0$ are $\pm\tan\frac{\pi}{7}, \pm\tan\frac{2\pi}{7}, \pm\tan\frac{4\pi}{7}$.

$t^6 - 21t^4 + 35t^2 - 7 = (t - \tan\frac{\pi}{7})(t - \tan\frac{2\pi}{7})(t - \tan\frac{4\pi}{7})(t + \tan\frac{\pi}{7})(t + \tan\frac{2\pi}{7})(t + \tan\frac{4\pi}{7})$

Let $x = t^2$, then the roots of $x^3 - 21x^2 + 35x - 7 = 0$ are $\tan^2\frac{\pi}{7}, \tan^2\frac{2\pi}{7}, \tan^2\frac{4\pi}{7}$.

By long division,

$t^6 - 21t^4 + 35t^2 - 7 = \left(t^3+\sqrt{7}t^2-7t+\sqrt{7}\right)\left(t^3-\sqrt{7}t^2-7t-\sqrt{7}\right)$

$\therefore x^3+\sqrt{7}x^2-7x+\sqrt{7}=0$ has roots $\alpha, \beta, \gamma$; where they are three out of $\pm\tan\frac{\pi}{7}, \pm\tan\frac{2\pi}{7}, \pm\tan\frac{4\pi}{7}$.

$\alpha + \beta + \gamma = -\sqrt{7} < 0$ …… (1)

$\alpha\beta + \beta\gamma + \alpha\gamma = -7 < 0$ ……(2)

$\alpha\beta\gamma = -\sqrt{7} < 0$ ……(3)

From (3), either two positive roots and one negative root or three negative roots.

If all roots are negative, i.e. $\alpha < 0, \beta < 0, \gamma < 0$

then (2) implies $-7 = \alpha\beta + \beta\gamma + \alpha\gamma > 0$, which is a contradiction

Without loss of generality, assume $\alpha > 0, \beta > 0, \gamma < 0$

Replace $t$ by $-t$ in the first factor gives $\left(-t^3+\sqrt{7}t^2+7t+\sqrt{7}\right) = -\left(t^3-\sqrt{7}t^2-7t-\sqrt{7}\right)$

$\therefore$ The roots of the second factor $\left(t^3-\sqrt{7}t^2-7t-\sqrt{7}\right) = 0$ is $-\alpha, -\beta, -\gamma$.

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = 2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} \div \left(2\sin\frac{\pi}{7}\right) = 2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} \div \left(4\sin\frac{\pi}{7}\right)$$

$$= 2\sin\frac{4\pi}{7}\cos\frac{4\pi}{7} \div \left(8\sin\frac{\pi}{7}\right) = \sin\frac{8\pi}{7} \div \left(8\sin\frac{\pi}{7}\right) = -\sin\frac{\pi}{7} \div \left(8\sin\frac{\pi}{7}\right) = -\frac{1}{8}$$

To find $-8\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}$.

Consider $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \Rightarrow \frac{z^7-1}{z-1} = 0, z \neq 1$

$z^7 = \cos 2k\pi + i \sin 2k\pi$, $k = -3, -2, -1, 1, 2, 3$

$z = \cos\theta + i\sin\theta$ or $z = \cos\theta - i\sin\theta$, where $\theta = \frac{2k\pi}{7}$, $k = 1, 2, 3$

Quadratic factors are $(z - \cos\theta + i\sin\theta)(z - \cos\theta - i\sin\theta) = z^2 - 2z\cos\theta + 1$

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = (z^2 - 2z\cos\frac{2\pi}{7} + 1)(z^2 - 2z\cos\frac{4\pi}{7} + 1)(z^2 - 2z\cos\frac{6\pi}{7} + 1)$

Put $z = 1$: $7 = (2 - 2\cos\frac{2\pi}{7})(2 - 2\cos\frac{4\pi}{7})(2 - 2\cos\frac{6\pi}{7}) = 8(1 - \cos\frac{2\pi}{7})(1 - \cos\frac{4\pi}{7})(1 - \cos\frac{6\pi}{7})$

$\frac{7}{8} = \left(1 - 1 + 2\sin^2\frac{\pi}{7}\right)\left(1 - 1 + 2\sin^2\frac{2\pi}{7}\right)\left(1 - 1 + 2\sin^2\frac{3\pi}{7}\right)$

$\frac{7}{8} = 8\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2$ $\quad (\because \sin^2\frac{3\pi}{7} = \sin^2\frac{4\pi}{7})$

$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = \pm\frac{\sqrt{7}}{8}$

$\because \sin\frac{\pi}{7} > 0$, $\sin\frac{2\pi}{7} > 0$ and $\sin\frac{4\pi}{7} > 0$

$\therefore \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = \frac{\sqrt{7}}{8} \Rightarrow -8\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = -\sqrt{7}$

$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{4\pi}{7} = \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} \div \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\sqrt{7}$

$\therefore$ The roots of $x^3 + \sqrt{7}x^2 - 7x + \sqrt{7} = 0$ are $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, $\tan\frac{4\pi}{7}$.